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**OPTIMUM PERFORMANCE OF HOVERING ROTORS**

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## OPTIMUM PERFORMANCE OF HOVERING ROTORS

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### SUMMARY

A theory for the optimum performance of a rotor hovering out of ground effect is developed. The performance problem is formulated using general momentum theory for an infinitely bladed rotor, and the effect of a finite number of blades is estimated. The analysis takes advantage of the fact that a simple relation exists between the radial distributions of static pressure and angular velocity in the ultimate wake, far downstream of the rotor, since the radial velocity vanishes there. This relation permits the establishment of an optimum performance criterion in terms of the ultimate wake velocities by introducing a small local perturbation of the rotational velocity and requiring the resulting ratio of thrust and power changes to be independent of the radial location of the perturbation. This analysis fully accounts for the changes in static pressure distribution and axial velocity distribution throughout the wake as the result of the local perturbation of the rotational velocity component. This improvement over earlier theories is shown to have finite contributions to the optimum distributions of circulation and inflow as well as to the rotor performance.

The velocities in the ultimate wake are related to the circulation and inflow distributions at the rotor disk using a constant contraction ratio for

the wake. Optimum distributions of the circulation and the inflow, together with the performance characteristics are presented for sixteen values of the thrust coefficient ranging from 0.001 to 0.050.

## I. INTRODUCTION

The development of a suitable theory for predicting the optimum performance of a rotor hovering out of ground effect (OGE) has long been a central problem of rotorcraft aerodynamics. Rotorcraft typically are required to be able to hover OGE when occasion demands. Since the power required for a rotor to develop a given amount of thrust is the greatest during hover OGE, the payload capability of the rotorcraft is often limited by the hovering performance of the rotor OGE. For a modern rotorcraft, the payload is likely to be  $1/5$  to  $1/4$  of the gross weight at take off. Consequently, a rotor which develops a thrust 5% less than the maximum attainable while hovering OGE will lead to a deficiency in payload capability of 20% to 25%. It is thus of great practical importance to study the performance of a rotor hovering OGE. The basic task in such a performance study is the determination of a distribution of inflow velocity, over a given rotor disk, that leads to a minimum amount of power expenditure for a given amount of thrust. This information, when used in conjunction with the well known blade-element theory, permits the prediction of the performance limit and the required radial distribution of circulation in order to approach this limit. It thus provides a rational basis for blade design.

A hovering rotor can be considered as a special case of a propeller, with zero advance velocity. Marine and aircraft engineers have long been concerned

with propeller aerodynamics. In the majority of situations of interest to propeller aerodynamicists, however, the propeller-induced velocity is small in comparison with the propeller's velocity of advance. As a result, propeller performance theories customarily make use of simplifications that are valid only if the induced velocity is small. These theories therefore usually do not go over to rotor theories in a straightforward manner and rotor performance predictions based on generalized propeller theories do not always correlate well with experimental data (Ref. 1).

In recent years, the search for better methods of predicting rotor performance in hover yielded a copious volume of literature on the subject. Several recent articles contain comprehensive bibliography (Ref. 2) of the earlier work as well as review and reassessment (Ref. 1) of "classical" theories. The brief discussion given below provides a summary account of previous theories and present efforts. Only a few pertinent articles are referred to here. In many cases, a large number of articles exist which are based on essentially the same approach. Only one representative article is then mentioned.

The earliest theory on propeller performance was the axial momentum theory, based on the actuator disk (infinitely-bladed propeller) concept by Rankine (Ref. 3) and Froude (Ref. 4). Within the context of this theory, the propeller operates without any frictional drag on the blade and induces no rotation in the slipstream. Since the flow upstream of the propeller disk is irrotational, the tangential velocity component at the disk is zero, and it follows from the Kutta-Joukowski theorem that the disk cannot be subjected to a lift force. Disregarding this inconsistency, two well known conclusions were obtained: first, an optimum propeller is one with a uniform axial

velocity at the propeller disk, and second, when applied to a hovering rotor, the minimum power coefficient is equal to the thrust coefficient to the  $3/2$  power divided by the square root of 2. The theory, however, provides no information on the radial distribution of circulation over the disk and consequently no indication as to how the blades should be designed for optimum performance.

In general, the energy loss due to the rotational motion in the slipstream of a propeller is small. The neglect of the slipstream rotation was therefore thought to be justified for the purpose of predicting the performance of propellers. The actual performance of propellers, however, generally does not meet the expectation of the axial momentum theory even when the various loss factors, not including the slipstream rotation, are taken into account. This deficiency is explained by the general momentum theory, which incorporates a procedure to account for some of the effects of slipstream rotation. One of the major conclusions of the general momentum theory is that the slipstream rotation, though small in terms of the energy content, exerts an important influence on the optimum distribution of inflow velocity over the propeller disk. A detailed presentation of the general momentum theory is given in Ref. 5 which also contains a presentation of efforts directed toward the study of helicopter rotors. The general momentum theory does provide information on the optimum distribution of circulation over the disk. The solutions as given in Ref. 5, however, are approximate and only partially account for the effects of slipstream rotation.

The theory which represents the current state of the art for routine calculation of propeller performance is the vortex theory. The basic vortex theory represents the wake trailing the blades by a distribution of concentric

cylindrical vortex sheets. These vortex sheets describe the radial variation of the blade circulation and thus imply an infinite number of blades. The effect of a finite number of blades is obtained approximately by Prandtl (Ref. 6) using a tip loss factor. An improved analysis by Goldstein (Ref. 7) represents the trailing vortex sheets, for optimum performance, by discrete helicoidal surfaces of constant helix angle and moving as rigid surfaces. Goldstein's work formed the basis of much of the subsequent analyses by Lock, Theodorsen, Lerbs, etc., (Refs. 8, 9, 10, and 11) that proved to be sufficiently accurate for predicting propeller performance. The use of the vortex theory for predicting hovering rotor performance, however, yielded overly optimistic results (Ref. 1). Reviews of the vortex theory suggest that the wake contraction, the slipstream rotation, and the associated non-uniform inflow are the factors that contribute most to the inability of the vortex theory to predict the hovering rotor performance accurately. This contention is supported by recent results obtained from the numerical computation of the entire rotor-induced flow field. Before discussing the numerical approach, it is to be noted that the work of Lerbs (Ref. 11) extends the vortex theory to heavily-loaded propellers where the effect of wake contraction is important. Lerbs' work, though well known in marine engineering, has not been extended and applied to the study of rotor performance. As is the case of the general momentum theory, the work of Lerbs is based on certain simplifying assumptions that cannot be justified for a hovering rotor. The modification of Lerbs' analysis for rotor applications is by no means straightforward.

The continuing requirements of higher forward flight speed and larger rotorcraft, coupled with the need to keep the rotor reasonably small from weight and operational flexibility considerations, led to the development of

rotors with higher rotational speed and disk loading. This resulted in increased inaccuracy of the existing theories in predicting the hovering performance. The most recent efforts to overcome this difficulty is the development of numerical methods for the integration of the Biot-Savart law; the use of which permits the computation of the entire flow field induced by the rotor. It is noted that calculation of the rotor performance requires only a knowledge of the inflow velocity over the propeller disk. The quadrature of the Biot-Savart law, in order to establish the inflow velocity components at the disk, however, requires the computation of the entire vortex system at and trailing the disk. Since the computation requires an iterative procedure and the number of data points involved in each iteration is large, the approach is time-consuming even by modern computational standards. To solve a problem where the propeller geometry, rather than the blade circulation distribution, is specified, an additional iterative procedure is needed to establish the circulation distribution corresponding to the specified propeller geometry. Further, to obtain the propeller geometry that would give the optimum performance, a parametric study involving a large number of geometries must be treated. For this reason, although some success has been reported in the development of the Biot-Savart law approach, the utility of this approach for design purposes is limited at the present.

The purpose of this report is twofold, (a) to present a simple theory, together with numerical results, for rotor performance and (b) to clarify certain features of the performance problem pertinent to rotors hovering OGE. The theory presented here is similar to the general momentum theory described in Ref. 5 in that the rotor is represented by an actuator disk. It is therefore subjected to some of the limitations of the general momentum theory. In

particular, the correction factor for finite number of blades must be estimated separately, for example, by an extension of Lerbs' (Ref. 10) method for heavily loaded propellers. The present method, however, is more complete in that it fully accounts for the effects of slipstream rotation. This improvement, which is not important in the case of a lightly loaded propeller, has finite contributions to thrust and power requirements of a rotor. The use of the present theory permits the calculation, in a straightforward manner, of the distribution of circulation and inflow velocity over the rotor disk for the infinitely bladed case. These distributions are, along with the figure of merit and the optimum power coefficient, presented for several values of the thrust coefficient.



# SYMBOLS

$a = u_1/\Omega R_1$ , dimensionless axial velocity in the ultimate wake

A value of "a" at the rim of the ultimate wake

$b = \omega_1/\Omega$ , dimensionless angular velocity in the ultimate wake

B value of "b" at the rim of the ultimate wake

H total head of the fluid

$H_0$  total head of the ambient fluid

$H_1$  total head of the fluid in the ultimate wake

$k = r_1/r$ , local contraction ratio

$K = R_1/R$ , overall contraction ratio

M figure of merit, Eq. (51)

n a constant in the optimum performance criterion, Eq. (32)

$N = n/R_1$ , a dimensionless constant

$N_b$  number of blades

p static pressure of the fluid immediately upstream of the rotor disk

$p_0$  static pressure of the ambient fluid

$p_1$  static pressure of the fluid in the ultimate wake

$p'$	static pressure jump across the rotor disk
$P$	power expended by the rotor
$P_c$	power coefficient, Eq. (50)
$Q$	torque on the rotor
$r$	radial position on the rotor disk
$r_1$	radial position in the ultimate wake
$R$	radius of the rotor
$R_1$	radius of the ultimate wake
$S$	area of the rotor disk
$S_1$	area of the ultimate wake
$T$	thrust developed by the rotor
$T_c$	thrust coefficient, Eq. (49)
$u$	axial velocity at the disk
$u_1$	axial velocity in the ultimate wake
$x = r_1/R_1$	dimensionless radial position in the ultimate wake
$y = r/R$	dimensionless radial position on the disk
$\Gamma$	circulation at the rotor disk

$\Gamma_b$	circulation of each blade
$\omega$	angular velocity of the fluid immediately downstream of the rotor disk
$\omega_1$	angular velocity of the fluid in the ultimate wake
$\Omega$	angular velocity of the rotor
$\rho$	density of the fluid

## II. FORMULATION OF THE HOVERING ROTOR PROBLEM

The equations governing the performance of hovering rotors are derived below within the context of the general momentum theory. The general flow features are shown in Fig. 1a. The rotor is represented by an actuator disk of radius  $R$  and driven to rotate about its axis at a constant angular velocity  $\Omega$  in an incompressible, inviscid fluid. The flow is steady and symmetric about the axis of the actuator disk. The rotational component of the velocity is, however, non-zero in the slipstream. The fluid outside the slipstream is irrotational and is at rest at infinity. The total head of the fluid outside the slipstream is therefore equal to the static pressure of the fluid far upstream and is a constant. The actuator disk imparts an increase in total head to the fluid passing through the disk in the form of an abrupt static pressure increase and a change in the angular velocity of the fluid from the zero angular momentum value immediately upstream to some finite value immediately downstream of the disk. The law of mass conservation requires the axial velocity component to be continuous across the disk. The radial velocity component is taken to be continuous across the disk within the context of the general momentum theory.

The slipstream, in which the flow is rotational and possesses a higher total head than the flow outside, contracts downstream of the disk and forms an ultimate wake far downstream of the disk. In the ultimate wake, the radial velocity component vanishes and the flow properties are independent of the axial coordinate. Let the flow through the annular element  $dS = 2\pi r dr$  at radius  $r$  at the disk, shown in Fig. 1a pass through the annular element  $dS_1 = 2\pi r_1 dr_1$  in the ultimate wake. The subscripts "1" designate flow

conditions in the ultimate wake. The unsubscripted variables designate the corresponding flow conditions immediately downstream of the disk.

The law of mass conservation states that

$$u_1 r_1 dr_1 = u r dr \quad (1)$$

The tangential component of the equation of motion requires the angular momentum of the fluid to be constant along streamtubes in the slipstream.

Thus one has

$$\omega_1 r_1^2 = \omega r^2 \quad (2)$$

The radial-component of the equation of motion requires the radial pressure gradient in the ultimate wake to be balanced by the centrifugal force on the fluid, since the radial velocity component is zero in the ultimate wake. Thus

$$\frac{dp_1}{dr_1} = \rho \omega_1^2 r_1 \quad (3)$$

Bernoulli's equation, applied to the flow upstream of the disk gives

$$H_o = p_o = p + \frac{1}{2} \rho (u^2 + v^2) \quad (4)$$

where the subscripts "o" designate flow conditions far upstream of the disk and  $p$  is the static pressure of fluid immediately upstream of the disk.

The total head immediate downstream of the disk is

$$H = p + p' + \frac{1}{2} \rho (u^2 + v^2 + \omega^2 r^2) \quad (5)$$

where  $p'$  is the pressure jump across the disk and  $\omega$  is the angular velocity of the fluid immediately downstream of the disk. Applying Bernoulli's equation to the flow downstream of the disk gives

$$\begin{aligned} H_1 &= p + p' + \frac{1}{2}\rho (u^2 + v^2 + \omega^2 r^2) \\ &= p_1 + \frac{1}{2}\rho (u_1^2 + \omega_1^2 r_1^2) \end{aligned} \quad (6)$$

Equation (5) and (6) gives

$$p' = (H_1 - H_0) - \frac{1}{2}\rho \omega^2 r^2 \quad (7)$$

The element of torque of the disk is equal to the flux of angular momentum imparted to the fluid passing through the annular disk element  $dS$ . Thus

$$dQ = 2\pi\rho u\omega r^3 dr \quad (8)$$

The element of power expended by the disk therefore is

$$\begin{aligned} dP &= \Omega dQ \\ &= 2\pi\rho\Omega u\omega r^3 dr \end{aligned} \quad (9)$$

This element of power is also equal to the increase in total head per unit time imparted to the fluid passing through the annular disk element, i.e.,

$$dP = 2\pi(H_1 - H_0)u r dr$$

Consequently

$$H_1 - H_0 = \rho\Omega\omega r^2 = \rho\Omega\omega_1 r_1^2 \quad (10)$$

Putting Eq. (10) into Eq. (7) yields

$$p' = \rho \left( \Omega - \frac{1}{2} \omega \right) \omega r^2 \quad (11)$$

The element of thrust developed by the disk is

$$\begin{aligned} dT &= p' dS \\ &= 2\pi\rho \left( \Omega - \frac{1}{2} \omega \right) \omega r^3 dr \end{aligned} \quad (12)$$

The total thrust developed and the total power expended by the entire disk are expressible in terms of the velocities at the disk by using Eqs. (9) and (12):

$$P = 2\pi\rho\Omega \int_0^R u\omega r^3 dr \quad (13)$$

and

$$T = 2\pi\rho \int_0^R \left( \Omega - \frac{1}{2} \omega \right) \omega r^3 dr \quad (14)$$

In terms of the velocities in the ultimate wake, the total power is, using Eqs. (1), (2), and (13),

$$P = 2\pi\rho\Omega \int_0^{R_1} u_1 \omega_1 r_1^3 dr_1 \quad (15)$$

An expression for the total thrust in terms of the wake properties may be obtained by specializing the expression for a propeller in forward motion given in Ref. 5 to the case of zero advance velocity:

$$T = 2\pi \left[ \rho \int_0^{R_1} u_1^2 r_1 dr_1 - \int_0^{R_1} (p_0 - p_1) r_1 dr_1 \right] \quad (16)$$

Although the derivation presented in Ref. 5 requires certain assumptions not appropriate for the hovering rotor case, it is shown in Appendix A of this report that Eq. (16) is valid for the hovering rotor.

Using Eqs. (4), (6), and (10), one obtains

$$p_0 - p_1 = \frac{1}{2} \rho u_1^2 - \rho (\Omega - \frac{1}{2} \omega_1) \omega_1 r_1^2 \quad (17)$$

Equation (16) can therefore be re-written as

$$T = 2\pi \rho \int_0^{R_1} \left[ \frac{1}{2} u_1^2 + \left( \Omega - \frac{1}{2} \omega_1 \right) \omega_1 r_1^2 \right] r_1 dr_1 \quad (18)$$

Differentiating Eq. (17) with respect to  $r_1$  gives, after using Eq. (3) and re-arranging terms

$$\begin{aligned} \frac{1}{2} \frac{du_1^2}{dr_1} &= (\Omega - \omega_1) \frac{d(\omega_1 r_1^2)}{dr_1} \\ &= (\Omega - \omega_1) \left( r_1^2 \frac{d\omega_1}{dr_1} + 2\omega_1 r_1 \right) \end{aligned} \quad (19)$$

At the rim of the ultimate wake,  $r_1 = R_1$  and  $p_0 = p_1$ . Equation (17) thus gives

$$u_1^2(R_1) = [2\Omega - \omega_1(R_1)] \omega_1(R_1) R_1^2 \quad (20)$$



### III. CRITERION FOR OPTIMUM PERFORMANCE

In the ultimate wake where the radial velocity vanishes, the simple relationship between the pressure and the angular velocity, Eq. (3), makes it possible to express the axial velocity  $u_1$  as a function of the angular velocity  $\omega_1$ ; i.e., if  $\omega_1(r_1)$  is specified, then  $u_1(r_1)$  follows by the quadrature of Eq. (19). Such a simple relationship does not exist at the disk, where the radial velocity component is non-zero. Consequently, in deriving a criterion for optimum performance, it has not been possible to consider the flow conditions at the disk directly. In the following analysis, a criterion for optimum performance is established in terms of the velocity distributions in the wake. These velocity distributions will subsequently be related to the velocity distributions at the rotor disk and to the thrust and power coefficients of the rotor.

Consider a given distribution of angular velocity  $\omega_1(r_1)$ . Let this distribution be perturbed to  $\omega_1(r_1) + \Delta\omega_1$ , with  $\Delta\omega_1$  given by

$$\Delta\omega_1 = \begin{cases} 0 & , \quad 0 < r_1 < \zeta \\ \epsilon & , \quad \zeta < r_1 < \zeta + \Delta\zeta \\ 0 & , \quad \zeta + \Delta\zeta < r_1 < R_1 \end{cases} \quad (21)$$

where  $\epsilon \ll \omega_1(\zeta)$  and  $\Delta\zeta \ll \zeta$ . As a result of this perturbation in  $\omega_1$ , there exist perturbations in the pressure and the axial velocity in the ultimate wake as shown in Fig. 1b. From Eq. (3), one has

$$p_1 = p_0 - \int_{r_1}^{R_1} \rho \omega_1^2 \xi d\xi \quad (22)$$

Thus, to the first order in  $\Delta\omega_1/\omega_1$ , the perturbation in  $p_1$ , which results

from the perturbation in  $\omega_1$ , is

$$\Delta p_1 = - 2\rho \int_{r_1}^{R_1} \omega_1 \Delta \omega_1 \xi d\xi \quad (23)$$

Placing Eq. (21) in Eq. (23) gives

$$\Delta p_1 = \begin{cases} - 2\rho \omega_1(\xi) \xi(\epsilon \Delta \xi) & 0 < r_1 < \xi \\ - 2\rho \omega_1(\xi) \xi[\epsilon(\xi + \Delta \xi - r_1)] & \xi < r_1 < \xi + \Delta \xi \\ 0 & \xi + \Delta \xi < r_1 < R_1 \end{cases} \quad (24)$$

where only the lowest order term in  $\epsilon/\omega_1$  and in  $\Delta \xi/\xi$  are kept in each interval of interest.

Using Eq. (17), the perturbation in  $u_1$  can be expressed as

$$\rho u_1 \Delta u_1 = - \Delta p_1 + \rho(\Omega - \omega_1) r_1^2 \Delta \omega_1 \quad (25)$$

Thus from Eqs. (21) and (24)

$$\rho u_1 \Delta u_1 = \begin{cases} 2\rho \omega_1(\xi) \xi(\epsilon \Delta \xi) & 0 < r_1 < \xi \\ 2\rho \omega_1(\xi) \xi[\epsilon(\xi + \Delta \xi - r_1)] + \rho[\Omega - \omega_1(\xi)] \xi^2 \epsilon & \xi < r_1 < \xi + \Delta \xi \\ 0 & \xi + \Delta \xi < r_1 < R_1 \end{cases} \quad (26)$$

Using Eqs. (16) and (15), the changes in thrust and in power due to the perturbation in  $\omega_1$  are

$$\Delta T = 2\pi \int_0^{R_1} [2\rho u_1 \Delta u_1 + \Delta p_1] r_1 dr_1 \quad (27)$$

and

$$\Delta P = 2\pi\rho\Omega \int_0^{R_1} (u_1 \Delta\omega_1 + \omega_1 \Delta u_1) r_1^3 dr_1 \quad (28)$$

Using Eqs. (21), (24), and (26), one obtains

$$\Delta T = 2\pi\rho[2\Omega - \omega_1(\zeta)] \zeta^3(\epsilon\Delta\zeta) \quad (29)$$

and

$$\Delta P = 2\pi\rho\Omega \left\{ u_1 \zeta^3 + \frac{\omega_1(\zeta) \zeta^5 (\Omega - \omega_1)}{u_1} + 2\omega_1 \zeta \int_0^\zeta \frac{\omega_1}{u_1} \xi^3 d\xi \right\} (\epsilon\Delta\zeta) \quad (30)$$

Equations (29) and (30) may be combined to express  $\Delta P$  in the form

$$\Delta P = g(\zeta) \Delta T \quad (31)$$

where  $g(\zeta)$  is a function of  $\zeta$ , the location where the angular velocity perturbation is introduced. Suppose that an angular velocity perturbation is introduced at  $\zeta_1$ , with resulting changes in power and thrust  $\Delta P_1$  and  $\Delta T_1$ . Suppose a second angular velocity perturbation is introduced at  $\zeta_2$  with resulting changes in power and thrust  $\Delta P_2$  and  $\Delta T_2$ . By suitably selecting the values of  $\epsilon\Delta\zeta$  at  $\zeta_1$  and  $\zeta_2$ , the net change in thrust due to the two perturbations is made to zero, i.e.,  $\Delta T = \Delta T_1 + \Delta T_2 = 0$ . Then, using Eq. (31), the net change in power is

$$\Delta P = \Delta P_1 + \Delta P_2 = [g(\zeta_1) - g(\zeta_2)] \Delta T_1$$

Since  $\Delta T_1$  can be made either positive or negative by choosing the sign of " $\epsilon$ "

in Eq. (29), it is possible to obtain a negative value for  $\Delta P$  if  $g(\zeta_1) \neq g(\zeta_2)$ . In other words, it will be possible for the rotor (with fixed  $R$  and  $\Omega$ ) to develop a given amount of thrust while expending a smaller amount of power by altering the given angular velocity distribution  $\omega_1(r_1)$ . This distribution therefore is not optimum.

It is obvious, therefore, that the optimum distribution  $\omega_1(r_1)$  requires  $g(\zeta_1) = g(\zeta_2)$ . In other words, the criterion for optimum performance is that  $g(\zeta)$  is a constant. Let this constant be  $\Omega_n$ . From Eqs. (29) and (30), the optimum performance criterion is

$$nr_1^2(2\Omega - \omega_1) = u_1 r_1^2 + 2\omega_1 \int_0^{r_1} \frac{\omega_1}{u_1} \xi^3 d\xi + \frac{\omega_1}{u_1} r_1^4 (\Omega - \omega_1) \quad (32)$$

A differential form of this performance criterion is obtained by differentiating both sides of Eq. (32) with respect to  $r_1$ . The resulting equation and Eq. (19), both containing the derivatives of  $u_1$  and  $\omega_1$  with respect to  $r_1$ , permit each of the derivatives,  $\frac{du_1}{dr_1}$  and  $\frac{d\omega_1}{dr_1}$ , to be expressed as a function of  $u_1$ ,  $\omega_1$ , and  $r_1$ . In terms of dimensionless variables  $x = r_1/R_1$ ,  $a = u_1/\Omega R_1$ ,  $b = \omega_1/\Omega$ , and the dimensionless constant  $N = n/R_1$ , the two derivatives are

$$\frac{da}{dx} = \frac{2ab(1-b)[2a^2 - a(4-b)N + 2bx^2]x}{a^4 - 2a^3N - a^2b(1-2b)x^2 + b^2(1-b)^2x^4} \quad (33)$$

and

$$\frac{db}{dx} = \frac{2b[a^4 - a^3(2-b)N + a^2b(3-2b)x^2 - b^2(1-b)^2x^4]}{[a^4 - 2a^3N - a^2b(1-2b)x^2 + b^2(1-b)^2x^4]x} \quad (34)$$

At the rim of the slipstream,  $a$  and  $b$  are related by Eq. (20) which, in the

non-dimensional form, is

$$A^2 = B(2 - B) \quad (35)$$

where A and B are values of a and b respectively at  $x = 1$ .

Taking the upper limit of integration to be  $R_1$  in Eq. (32) and using Eq. (35), one obtains an expression for N:

$$N = \frac{B}{2-B} \left[ \frac{3-2B}{A} + 2 \int_0^1 \frac{b}{a} x^3 dx \right] \quad (36)$$

For any given value of B (or A), the system of equations (33) to (36) determine functions  $a(x)$  and  $b(x)$  that give optimum rotor performance as well as the value of N. For a given value of B, the value of A is determined by Eq. (35). The value of N depends on the functions  $a(x)$  and  $b(x)$  and is determined in an iterative procedure together with the functions  $a(x)$  and  $b(x)$ . For the first iteration, an estimated value of N is placed in Eqs. (33) and (34), which are solved for  $a(x)$  and  $b(x)$  using a fourth order Runge-Kutta method for simultaneous first order ordinary differential equations. The calculation begins at the rim,  $x = 1$ , where the boundary values A and B are known, and proceeds inward towards the axis,  $x = 0$ . The results are used to obtain a corrected value of N for the subsequent iteration. The iterative procedure continues until the quadrature of Eq. (36) yields a value of N differing insignificantly from the input value of N in Eqs. (33) and (34). The detailed computational procedures are presented in Appendix B.

#### IV. PERFORMANCE AND FLOW CONDITIONS AT THE ROTOR DISK

For a given radius of the ultimate wake,  $R_1$ , and a given rotor angular velocity  $\Omega$ , the optimum performance functions  $a(x)$  and  $b(x)$  specify the optimum flow field in the ultimate wake. The corresponding power expended and thrust developed by the rotor are determinant from Eqs. (15) and (18). The optimum circulation and the axial inflow velocity at the disk as well as the power and thrust coefficients and the figure of merit of the rotor can be computed based on the known functions  $a(x)$  and  $b(x)$ , provided that a functional relationship between  $r$  and  $r_1$  can be established. In theories of rotor and propeller performance, it is customary to obtain a differential form of the thrust element by differentiating equation (18):

$$dT = 2\pi\rho \left[ \frac{1}{2} u_1^2 + \left( \Omega - \frac{1}{2} \omega_1 \right) \omega_1 r_1^2 \right] r_1 dr_1 \quad (37)$$

Equating the right sides of Eqs. (12) and (37) gives

$$\frac{dr}{dr_1} = \frac{\left[ u_1^2 + (2\Omega - \omega_1) \omega_1 r_1^2 \right] r_1}{(2\Omega - \omega) \omega r^3} \quad (38)$$

Equations (2) and (38), together with the boundary condition

$$r = 0 \quad \text{at} \quad r_1 = 0 \quad (39)$$

determine a functional relationship between  $r$  and  $r_1$ , including the overall contraction ratio  $K = R_1/R$ , when the velocity distributions  $u_1$  and  $\omega_1$  in the wake are prescribed.

Equation (37) obviously is not the only expression that satisfies Eq. (18);

for any expression  $g(r_1)dr_1$  which gives  $\int_0^{R_1} g(r_1)dr_1 = 0$  can be added to the right side of Eq. (37), and the resulting expression still satisfies Eq. (18). In fact, it is shown in Appendix C that for  $\omega_1$  bounded and  $r(r_1)$  a one to one function, no solution of Eq. (38) exists which satisfies the boundary condition (39). A more detailed discussion is given in Ref. 15.

The function  $r(r_1)$  can be obtained as a part of the solution of the complete flow field between the disk and the ultimate wake. The differential equation governing the flow field is elliptic (Ref. 12) and it has been suggested (Ref. 13) that the exact contraction ratio can be obtained only by the complete solution of the governing equation. Such a complete solution constitutes a major task demanding a large amount of computational effort. Accordingly, in the present work, an approach which requires neither the use of Eq. (38) nor the solution for the entire flow field is developed. In this approach, the local contraction ratio  $k = r_1/r$  is taken to be independent of the radial position  $r$  (or  $r_1$ ).

It is recognized that in general the local contraction ratio depends on the radial position. The results of Ref. 10, however, indicate that for heavily loaded free-running propellers, the contraction ratio is nearly independent of the radial position, except near the axis of the propeller, where the dependence of the contraction ratio on the radial position has only small effects on the propeller's performance. Thus it is expected that the assumption of a constant contraction ratio is reasonable for rotor performance studies.

With  $k(r) = K$ , one has

$$r = r_1/K \quad \text{and} \quad dr = dr_1/K \quad (40)$$

Thus Eqs. (1) and (2) become

$$u = u_1 K^2 \quad (41)$$

and

$$\omega = \omega_1 K^2 \quad (42)$$

The value of K can be determined by equating the right sides of the two total thrust equations, (14) and (18). Using Eqs. (40) to (42) and rearranging terms, one obtains the following expression for K:

$$K = \left[ 1 + \frac{\int_0^{R_1} u_1^2 r_1 dr_1}{2\Omega \int_0^{R_1} \omega_1 r_1^3 dr_1} \right]^{-\frac{1}{2}} \quad (43)$$

Integrating by parts gives

$$\int_0^{R_1} u_1^2 r_1 dr_1 = \frac{U_1^2(R_1) R_1^2}{2} - \frac{1}{2} \int_0^{R_1} r_1^2 \frac{du_1^2}{dr_1} dr_1 \quad (44)$$

Using Eqs. (19) and (20) and integrating by parts yields

$$\int_0^{R_1} u_1^2 r_1 dr_1 = 2\Omega \int_0^{R_1} \omega_1 r_1^3 dr_1 \quad (45)$$

Putting Eq. (45) into Eq. (43) gives

$$K = 1/\sqrt{2} \quad (46)$$

which is the value predicted by the axial momentum theorem.

The axial inflow velocity and the circulation at the disk are related to



the axial and angular velocities in the ultimate wake by

$$u(r) = \frac{1}{2} u_1(r_1) \quad (47)$$

and

$$\Gamma(r) = 2\pi\omega_1(r_1)r_1^2 \quad (48)$$

The thrust and power coefficients are given by

$$T_c = \frac{T}{\pi R^2 \rho (\Omega R)^2} = \frac{1}{4} \int_0^1 [a^2 + (2 - b) bx^2] x dx \quad (49)$$

$$P_c = \frac{P}{\pi R^2 \rho (\Omega R)^3} = \frac{1}{2\sqrt{2}} \int_0^1 abx^3 dx \quad (50)$$

The figure of merit is given by

$$M = \frac{T_c^{3/2}}{\sqrt{2} P_c} \quad (51)$$

## V. RESULTS AND DISCUSSIONS

The optimum performance characteristics of the hovering rotor are presented in Table 1. The minimum power coefficient, the optimum figure of merit and the non-dimensional circulation at the disk edge are given for 16 values of the thrust coefficient between 0.001 and 0.050. This range of thrust coefficient was considered to bracket the normal operating range of hovering rotors.

The optimum radial distributions of circulation on the disk are presented

in Fig. 2 for the 16 values of the thrust coefficient. It can be seen that for small values of  $T_c$ , the optimum circulation is nearly independent of the radial position except near the axis. For higher values of  $T_c$ , the deviation of the optimum circulation from a constant value extends over a larger region and the magnitude of the deviation is greater.

The corresponding optimum distributions of the axial inflow velocity at the disk are presented in Fig. 3. It can be seen that a similar pattern exists for the axial inflow velocity as for the circulation. That is, the optimum axial inflow velocity is nearly uniform for small values of  $T_c$  except near the axis. The deviation of optimum axial inflow velocity from a constant value is more significant for the higher  $T_c$  values. The slope of the curves  $u(r)$  is zero at  $r = 0$  as expected.

In Ref. 5, an approximate solution for the hovering rotor problem is given based on the assumptions that the radial pressure gradient in the ultimate wake is negligible and that the non-dimensional velocities at the disk  $\tilde{u}(y) = \frac{u}{\Omega R}$  and  $\tilde{\omega}(y) = \frac{\omega}{\Omega}$  are

$$\tilde{u} = \frac{my^2}{m^2 + y^2} \quad (52)$$

and

$$\tilde{\omega} = \frac{2m^2}{m^2 + y^2} \quad (53)$$

where  $m$  is a constant and  $y = r/R$ . The approximate function  $\tilde{\omega}(y)$  and  $\tilde{u}(y)$  are compared to the present results for optimum performance in Figs. 4 and 5 for the case  $T_c = 0.01$ . A similar comparison for the circulation is given in Fig. 6. The comparisons show that the approximate results deviate signifi-

cantly from the present results, particularly in the inboard region of the disk. The deviations are more severe for larger values of  $T_c$ . The approximate results give excessively high value of  $\omega$  and zero value of  $u$  at the axis, as is shown to be incorrect in Appendix C. The present solution gives a moderate value of  $\omega$  and a finite value of  $u$  at the axis.

The variation of pressure in the ultimate wake is plotted against the radial position in Fig. 7, again for the case  $T_c = 0.01$ . As can be seen from Eq. (17), if the pressure gradient in the ultimate wake is neglected, as was done in Ref. 5, the axial velocity vanishes at the axis. If the pressure gradient is taken into account, as is in the present analysis, the axial velocity is finite at  $x = 0$ . The value of  $(p_o - p_1)/\rho(\Omega R_1)^2 = 0.0123$  at  $x_1 = 0$ , given in Fig. 7, corresponds to a value of  $u_1 = 0.157 \Omega R_1$ , which is comparable to the maximum axial velocity in the wake,  $0.207 \Omega R_1$ . The effect of the pressure gradient in the ultimate wake on the thrust developed by the rotor is represented by the second term on the right side of Eq. (16). The neglect of the pressure gradient would lead to, according to Fig. 7, a prediction of the thrust coefficient approximately 2% higher than the optimum for the case  $T_c = 0.01$ . More important is the fact that the optimum radial distributions of circulation and axial inflow velocity predicted by the approximate theory, which neglects the pressure gradient, contain significant errors.

In Fig. 8, the optimum power coefficient is presented as a function of the thrust coefficient. The dependence of the figure of merit on the thrust coefficient is shown in Fig. 9. The present result is compared with the approximate result of Ref. 5 and the ideal figure of merit. The figure shows that the approximate method of Ref. 5 predicts the rotor performance somewhat more

optimistically than the present theory. Both the approximate method and the present method predict performance substantially lower than ideal. The deviation from the ideal performance increases with increasing thrust coefficient.

The present results are primarily concerned with an infinitely bladed rotor. The effect of a finite number of blades may be determined, for example, by extending Lerbs' work (Ref. 10) on heavily loaded, free running propellers to hovering rotors. An estimate of the effect of number of blades on the optimum distribution of circulation is provided by modifying Prandtl's approximate method (Ref. 6) of calculating the "tip loss" factor and applying the result to the hovering rotor case. The procedure is outlined in Appendix D. The results are shown in Figure 10 for rotors with 2, 4, and 6 blades at  $T_c = 0.010$ . Based on these results, the corrected thrust coefficients for 2, 4, and 6 blades are respectively 0.00759, 0.00865, and .00906. The corresponding corrected power coefficients are .000547, 0.000624, and 0.000655. The figures of merit are 0.855, 0.911, and 0.931. In comparison, without the tip loss correction, the power coefficients for the infinitely bladed rotor are respectively (at  $T_c = 0.00759, 0.00865, \text{ and } 0.00906$ ) 0.000479, 0.000585, and 0.000627. The figures of merits are 0.975, 0.973, and 0.972.

## VI. CONCLUSIONS

A theory for the optimum performance of a rotor hovering out of ground effect is presented. The formulation of the problem is based on the general momentum theory of an actuator disk (infinitely bladed rotor) and makes use of the fact that in the ultimate wake, far downstream of the rotor disk, the radial component of the velocity vanishes and there exists a simple relation between the radial distribution of static pressure and that of rotational velocity component. The present analysis, which leads to an optimum performance criterion, is more complete than previous analyses in that the present work fully accounts for (1) the effect of slipstream rotation and (2) the existence of a radial pressure gradient in the ultimate wake.

Numerical results are presented for the radial distributions of circulation and the axial inflow velocity at the disk, the optimum power coefficient and the optimum figure of merit for sixteen values of the thrust coefficient covering the normal range of interest in rotorcraft application. Comparisons are made between previous approximate results and the present more exact results.

Several conclusions of the present study are summarized below:

1. The customarily accepted expression for the differential element of thrust in terms of the flow conditions in the ultimate wake, used in previous analyses to relate the wake flow to the flow at the disk, is inconsistent with the fact that a radial pressure gradient exists in the ultimate wake.
2. The existence of the radial pressure gradient in the ultimate wake has a significant effect on the optimum radial distribution of circulation and the optimum performance of the rotor in the range of thrust coefficient of

interest to rotorcraft application.

3. Numerical results obtained based on the present theory indicate that previous approximate theories yielded overly optimistic results on the performance of hovering rotors. The inaccuracy in the approximate results increases with increasing thrust coefficient. As has been noted by earlier investigators comparing experimental data with theoretical results, the existing theories generally overestimate the hovering rotor performance in the higher thrust coefficient range. Thus the correction factor based on the present theory may be utilized to improve the previous predictions.

4. With the assumption of uniform contraction, the contraction ratio is  $1/\sqrt{2}$ , the value predicted by the axial momentum theory.

5. The approach developed is potentially useful for predicting the optimum performance and the related distribution of circulation of heavily loaded propellers and rotors in very general types of axi-symmetric flow, such as the shrouded propeller and the ducted fan flows. The effect of finite number of blades can be determined separately by modifying available theories.

TABLE 1

## OPTIMUM PERFORMANCE CHARACTERISTICS

$T_c \times 10^2$	$P_c \times 10^3$	M	$B \times 10^2$
0.1	.02246	.9954	.2026
0.2	.06377	.9917	.4091
0.3	.1175	.9885	.6189
0.4	.1815	.9854	.8317
0.5	.2545	.9824	1.0473
0.6	.3355	.9796	1.2657
0.7	.4239	.9769	1.4866
0.8	.5193	.9743	1.710
0.9	.6213	.9717	1.9358
1.0	.7296	.9692	2.1640
1.5	1.3569	.9574	3.3393
2.0	2.1134	.9463	4.569
2.5	2.9865	.9359	5.8510
3.0	3.9682	.9259	7.1841
4.0	6.2379	.9086	10.0024
5.0	8.8958	.8887	13.0246

## APPENDIX A

### TOTAL THRUST IN TERMS OF THE WAKE PROPERTIES

Consider a spherical control volume of radius  $R$  centered at the center of the rotor disk. Let this control surface be divided into two parts,  $S_a$  and  $S_b$ ,  $S_a$  being the part of the control surface where the slipstream leaves the control volume and  $S_b$  being the remaining part of the control surface. The momentum theorem gives, for the present steady flow problem,

$$T = \int_{S_a} (p + \rho u^2) \bar{n} \cdot \bar{i} \, dS_a + \int_{S_b} (p + \rho u^2) \bar{n} \cdot \bar{i} \, dS_b \quad (A-1)$$

where  $p$  and  $u$  are the static pressure and the axial velocity of the fluid,  $\bar{n}$  is the unit outward normal and  $\bar{i}$  is the unit vector in the axial direction. At large distances from the rotor disk, the effect of the rotor on the fluid outside the slipstream is equivalent to that of a sink located at the center of the rotor disk. The strength of the sink is equal to the volume rate of flow passing through the disk and is finite. In the limit as  $R \rightarrow \infty$ , on  $S_b$  the magnitude  $w$  of the velocity vector goes to zero; i.e.,  $w \rightarrow \frac{C}{R^2}$  where  $C$  is a constant. Consequently  $(p_0 - p) = \frac{1}{2} \rho w^2 \rightarrow \frac{\rho C^4}{2R^4}$  on  $S_b$  as  $R \rightarrow \infty$ . The second integral in Eq. (A-1) thus gives  $-p_0(\pi R_1^2)$  as  $R \rightarrow \infty$ , and equation (A-1) becomes

$$T = 2\pi \int_0^{R_1} [(p_1 - p_0) + \rho u_1^2] r_1 \, dr_1 \quad (A-2)$$



## APPENDIX B

### COMPUTATIONAL PROCEDURES

Equations (33) and (34) constitute a system of two first order ordinary differential equations for the functions  $a(x, N)$  and  $b(x, N)$ . Equation (36) is an auxiliary equation relating  $N$  to the functions  $a(x, N)$  and  $b(x, N)$ . Equation (35) relates the values of  $a$  and  $b$  at  $x = 1$ . Thus, if either  $A = a(1)$  or  $B = b(1)$  is given, the functions  $a$  and  $b$  are obtainable from Eqs. (33), (34), and (36) by using an iterative method to establish the value of  $N$ .

For a selected value of  $B$ , the value of  $A$  is calculated using Eq. (35). The procedure for the  $i$ th iteration is as follows:

1. With a selected value of  $N^{(i)}$ , designated  $N_{in}^{(i)}$ , Eqs. (33) and (34) are solved for  $a^{(i)}(x)$  and  $b^{(i)}(x)$  using a fourth order Runge-Kutta method for simultaneous solution of the differential equations. The computation begins at  $x = 1$  where the values of  $a$  and  $b$  are known and proceeds inboard, at intervals of  $\Delta x = 0.01$ , to  $x = 0$ .

2. With the calculated  $a^{(i)}(x)$  and  $b^{(i)}(x)$ , a new value of  $N^{(i)}$ , designated  $N_{out}^{(i)}$ , is computed by numerical quadrature of the integral in Eq. (36) using a Newton-Cotes formula.

3. The difference  $F^{(i)} = N_{out}^{(i)} - N_{in}^{(i)}$  is calculated and compared to  $N_{in}^{(i)}$ . If the magnitude of  $F^{(i)}$  is less than  $10^{-8} N_{in}^{(i)}$ , then the computation is terminated and the functions  $a^{(i)}$  and  $b^{(i)}$  are accepted as the optimum function  $a(x)$  and  $b(x)$  for the given value of  $B$ . Otherwise a value of  $N_{in}^{(i+1)}$  is determined for the subsequent iteration.

The selection of the value  $N_{in}^{(1)}$  for the initial iteration is critical. As is shown in Fig. 3, the function  $a(x)$  is positive for optimum performance. If the value  $N_{in}^{(1)}$  is improperly selected, the calculated function  $a^{(1)}(x)$  may vanish at some points in the interval  $0 < x < 1$ . The numerical quadrature of the integral in Eq. (36) then becomes inaccurate. In actual computation, it was found that a value of  $N_{in}^{(1)}$  only slightly smaller than the correct value of  $N$  may lead to computational difficulties.

A reasonable value of  $N_{in}^{(1)}$  is obtained by using the form of  $a$  and  $b$  given by Glauert's approximate expressions, Eqs. (52) and (53). That is,

$$a = \frac{C_1 m x^2}{m^2 + x^2} \quad (B-1)$$

and

$$b = \frac{C_2 m^2}{m^2 + x^2} \quad (B-2)$$

To satisfy Eq. (35) for all values of  $m$ , one has  $C_1 = C_2 = 2$ . Placing (B-1) and (B-2) in Eq. (36) gives

$$N_{in}^{(1)} = \frac{\sqrt{B(2-B)} (3-B)}{(2-B)^2} \quad (B-3)$$

It was found that a linear iteration procedure, i.e., letting  $N_{in}^{i+1} = N_{out}^i$ , leads to very slow convergence. In the actual computation, for the second iteration, the value of  $N_{in}^{(2)}$  is taken to be  $0.98 N_{in}^{(1)}$ . For all subsequent iterations,  $N_{in}^{(i+1)}$  is obtained using the secant method, i.e.,

$$N_{in}^{(i+1)} = N^{(i)} - \left( \frac{F^{(i)}}{F^{(i)} - F^{(i-1)}} \right) (N_{in}^{(i)} - N_{in}^{(i-1)}) \quad (B-4)$$

From Eqs. (17) and (22), one obtains at  $x = 0$

$$a^2(0) = 2 \int_0^1 b^2 x dx \quad (B-5)$$

From Eq. (32), it can be shown that

$$b(0) = 2 - \frac{a}{N} \quad (B-6)$$

Equations (B-5) and (B-6) provide convenient means for checking the accuracy of the solutions.

In order to obtain the functions  $a(x)$  and  $b(x)$  for given values of the thrust coefficient, instead of given values of  $B$ , it is necessary to iterate for different values of  $B$ . This iteration is straightforward and involves no computational difficulty.

## APPENDIX C

### RESTRICTIONS DUE TO RADIAL PRESSURE GRADIENT IN THE WAKE

Equations (19) and (20) give the following equation:

$$u_1^2 = (2\Omega - \omega_1) \omega_1 r_1^2 + 2 \int_{r_1}^{R_1} \omega_1^2 r_1 dr_1 \quad (C-1)$$

Since  $\omega_1$  is bounded and non-negative, the first term on the right side of Eq. (C-1) approaches zero as  $r_1 \rightarrow 0$ . Thus  $u_1$  is non-zero at the axis  $r_1 = 0$ . The integral in Eq. (C-1) is positive and increases as  $r_1 \rightarrow 0$ . Hence there is a point S and a positive constant  $\kappa$  such that for  $r_1 < S$ ,

$$u_1^2 + (2\Omega - \omega_1) \omega_1 r_1^2 \geq \kappa \quad (C-2)$$

Consequently, from Eq. (38), one has for  $r_1 < S$

$$\frac{dr}{dr_1} \geq \frac{\kappa r_1}{(2\Omega - \omega) \omega r^3} \quad (C-3)$$

Since  $r(r_1)$  is a one to one function, neither  $\omega$  nor  $(2\Omega - \omega)$  can be negative for  $r_1 \leq S$ . Equation (C-3) may therefore be rewritten as

$$\frac{dr}{dr_1} \geq \frac{\kappa r_1}{2\Omega \omega r^3} = \frac{\kappa r_1}{2\Omega \omega_1 r_1^2 r} \geq \frac{\kappa}{2\Omega M r_1 r} \quad (C-4)$$

where M is a constant such that  $\omega_1 < M$  for  $r_1 < S$ . Let  $r'$  be given by

$$\frac{dr'}{dr_1} = \frac{\kappa}{2\Omega M r_1 r'} \leq \frac{dr}{dr_1} \quad (C-5)$$

and  $r' = r_1(S) \quad \text{at} \quad r_1 = S \quad (C-6)$

then, from Eq. (C-4), one has

$$r' \leq r \quad \text{for} \quad r_1 < S \quad (C-7)$$

However, Eqs. (C-5) and (C-6) give

$$r'^2 = r_1^2(S) - \frac{\kappa}{\Omega M} \ln(S/r_1) \quad (C-8)$$

Consequently  $r' = 0$  at  $r_1 = S \exp\left[-\frac{\Omega M r_1^2(S)}{\kappa}\right] > 0$ . Since  $r \leq r'$  for  $r_1 < S$ ,

one has  $r = 0$  at a point  $r_1 \geq S \exp\left[-\frac{\Omega M r_1^2(S)}{\kappa}\right] > 0$ . The condition  $r = 0$  at  $r_1 = 0$  therefore cannot be satisfied.

It is noted that, if the radial pressure gradient in the ultimate wake is neglected, as was done in Ref. 5, then the last term in Eq. (C-1) vanishes. Consequently,  $u_1 = 0$  at  $r_1 = 0$  and the condition  $r = 0$  at  $r_1 = 0$  can be satisfied. The consideration of the effects of the slipstream rotation, however, requires the pressure gradient term to be retained.

## APPENDIX D

### ESTIMATED TIP LOSSES BASED ON PRANDTL'S METHOD

Prandtl's approximate formula for the tip loss correction factor for the blade circulation is

$$F = \frac{2}{\pi} \cos^{-1} (e^{-f}) \quad (D-1)$$

where  $f = \frac{\pi g}{h}$ ,  $g$  being the distance from the rim of the trailing vortex sheet and  $h$  being the normal distance between two adjacent vortex sheets at the rim of the trailing vortex system.

Prandtl's formula was derived for propellers with finite velocity of advance and negligible slipstream contraction. The basic concept, however, has been utilized to study the hovering rotor problem (Ref. 14). Applying Eq. (D-1) to the ultimate wake, one has

$$g = R_1 - r_1$$

and

$$h = \frac{2\pi U_1 (R_1)}{N_b \Omega}$$

where  $N_b$  is the number of blades. Consequently,

$$f = \frac{N_b (1 - x)}{2A} \quad (D-2)$$

The total circulation of the rotor is therefore

$$N_b \Gamma_b = 2\pi\omega_1 r_1^2 F \quad (D-3)$$

The thrust and the power are, in terms of  $\Gamma_b$ ,

$$T = \rho N_b \int_0^R \Gamma_b r (\Omega - \frac{1}{2} \omega) dr \quad (D-4)$$

$$P = \rho N_b \Omega \int_0^R \Gamma_b r dr \quad (D-5)$$

Thus the thrust and power coefficients are

$$T_c = \int_0^1 C \left( 2 - \frac{\omega}{\Omega} \right) y dy \quad (D-6)$$

$$P_c = \int_0^1 \frac{C u}{\Omega R} y dy \quad (D-7)$$

where

$$C = \frac{N_b \Gamma_b}{2\pi \Omega R^2} \quad (D-8)$$

With the contraction factor  $k = \frac{1}{\sqrt{2}}$ , one has

$$y = x$$

$$\frac{u}{\Omega R} = \frac{1}{2\sqrt{2}} a$$

and

$$\frac{\omega}{\Omega} = \frac{1}{2} b$$

Thus, Eqs. (D-7) and (D-8) give

$$T_c = \int_0^1 c \left( 2 - \frac{b}{2} \right) y dy \quad (D-9)$$

and

$$P_c = \frac{1}{\sqrt{2}} \int_0^1 C y dy \quad (D-10)$$



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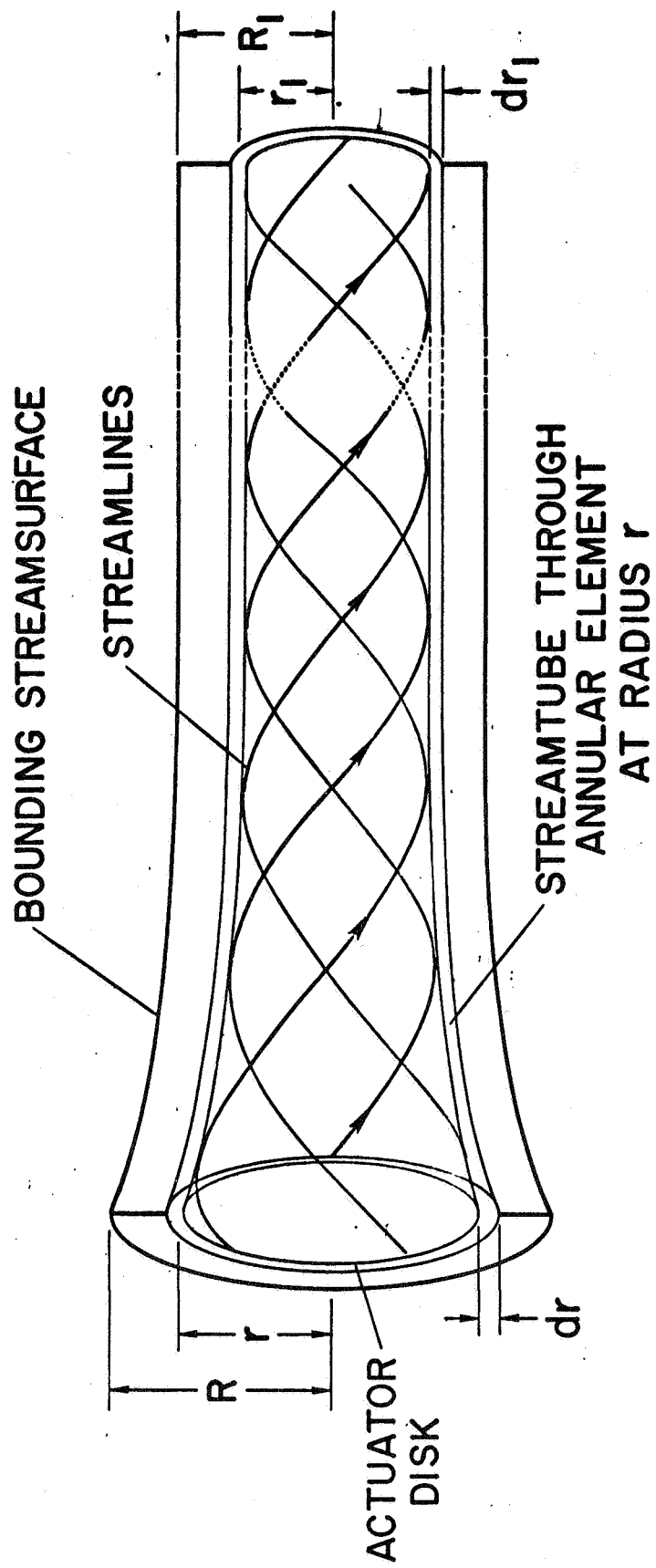


Figure 1 a General flow features

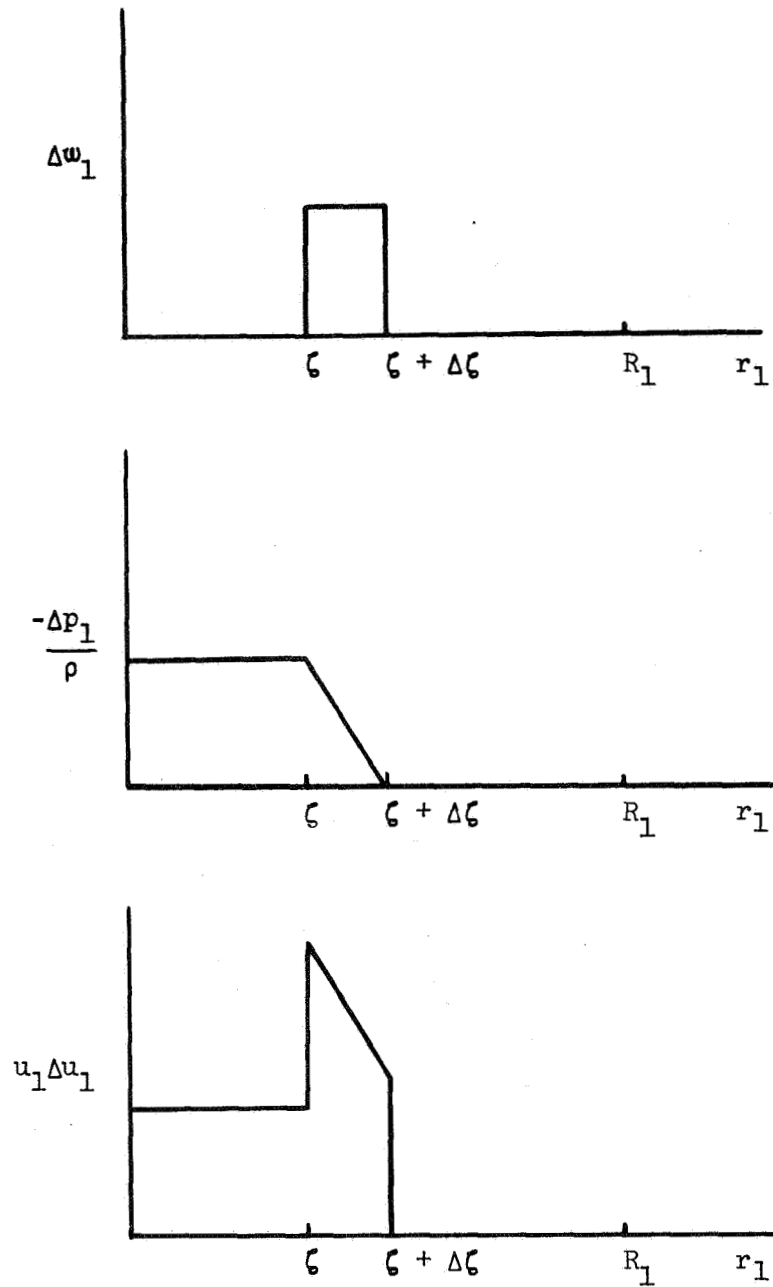


FIGURE 1b. ANGULAR VELOCITY PERTURBATION AND  
RESULTING PERTURBATIONS IN PRESSURE  
AND AXIAL VELOCITY

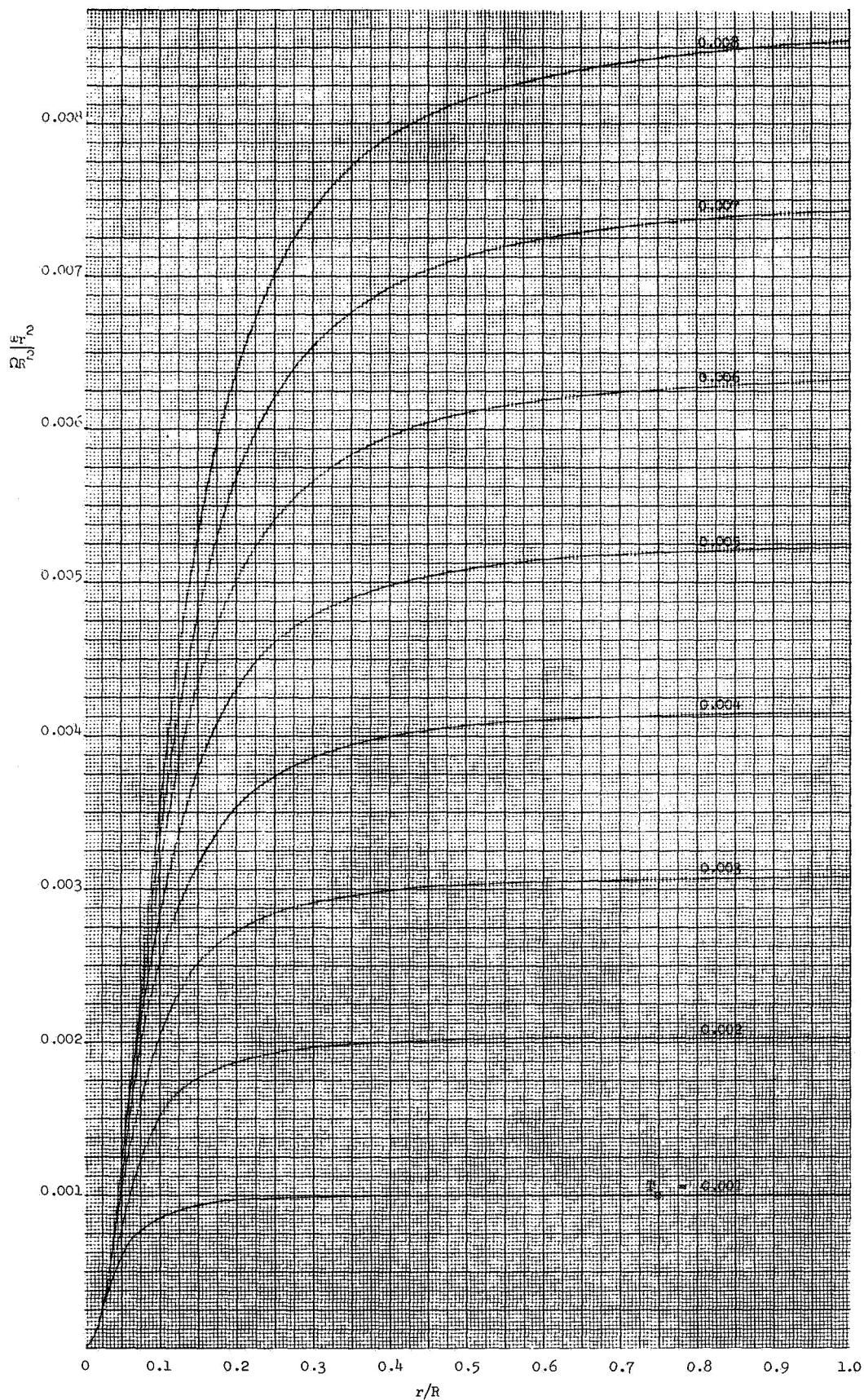


Figure 2(a).-- Optimum radial distribution of circulation for various values of thrust coefficient.

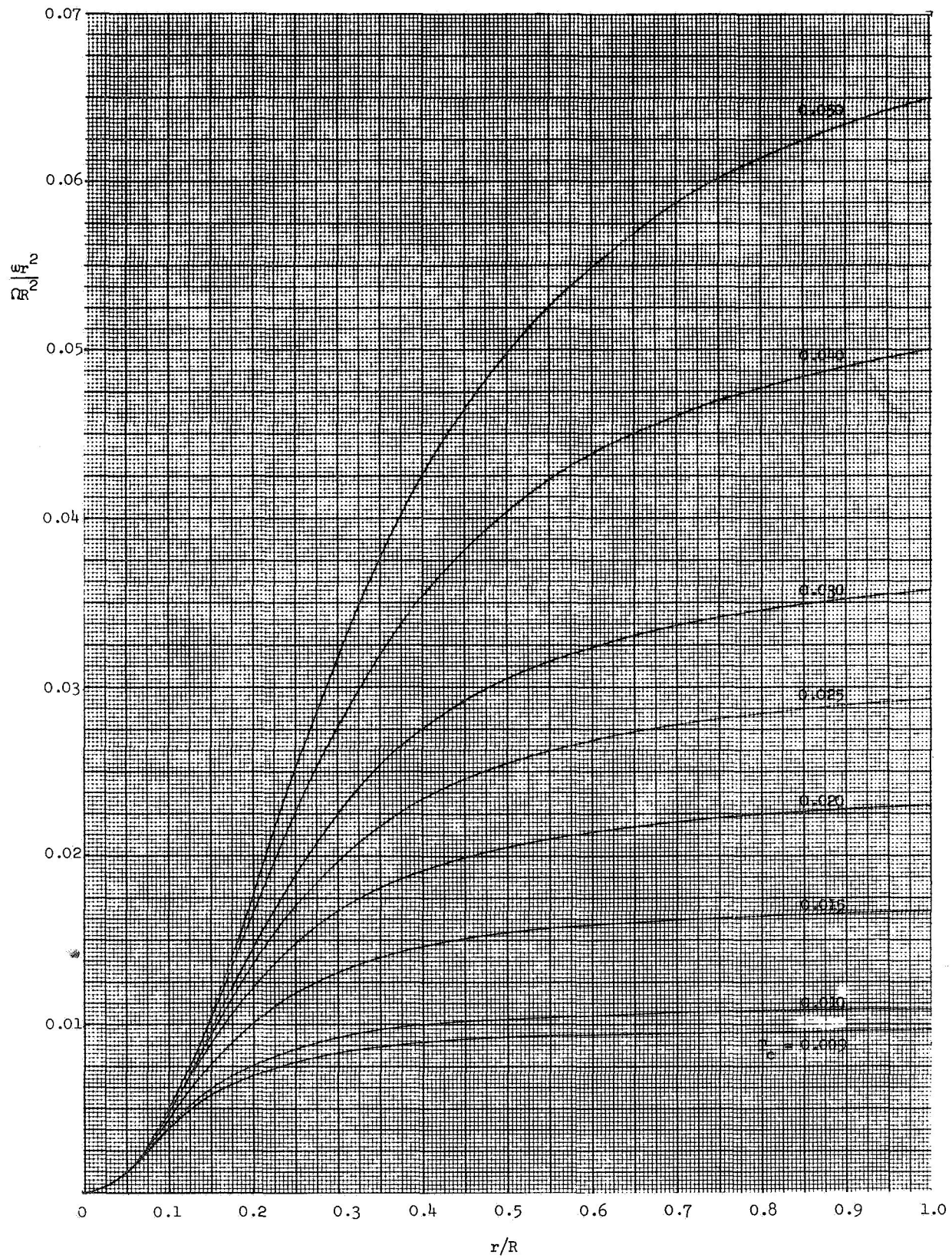


Figure 2(b).- Optimum radial distribution of circulation for various values of thrust coefficient.



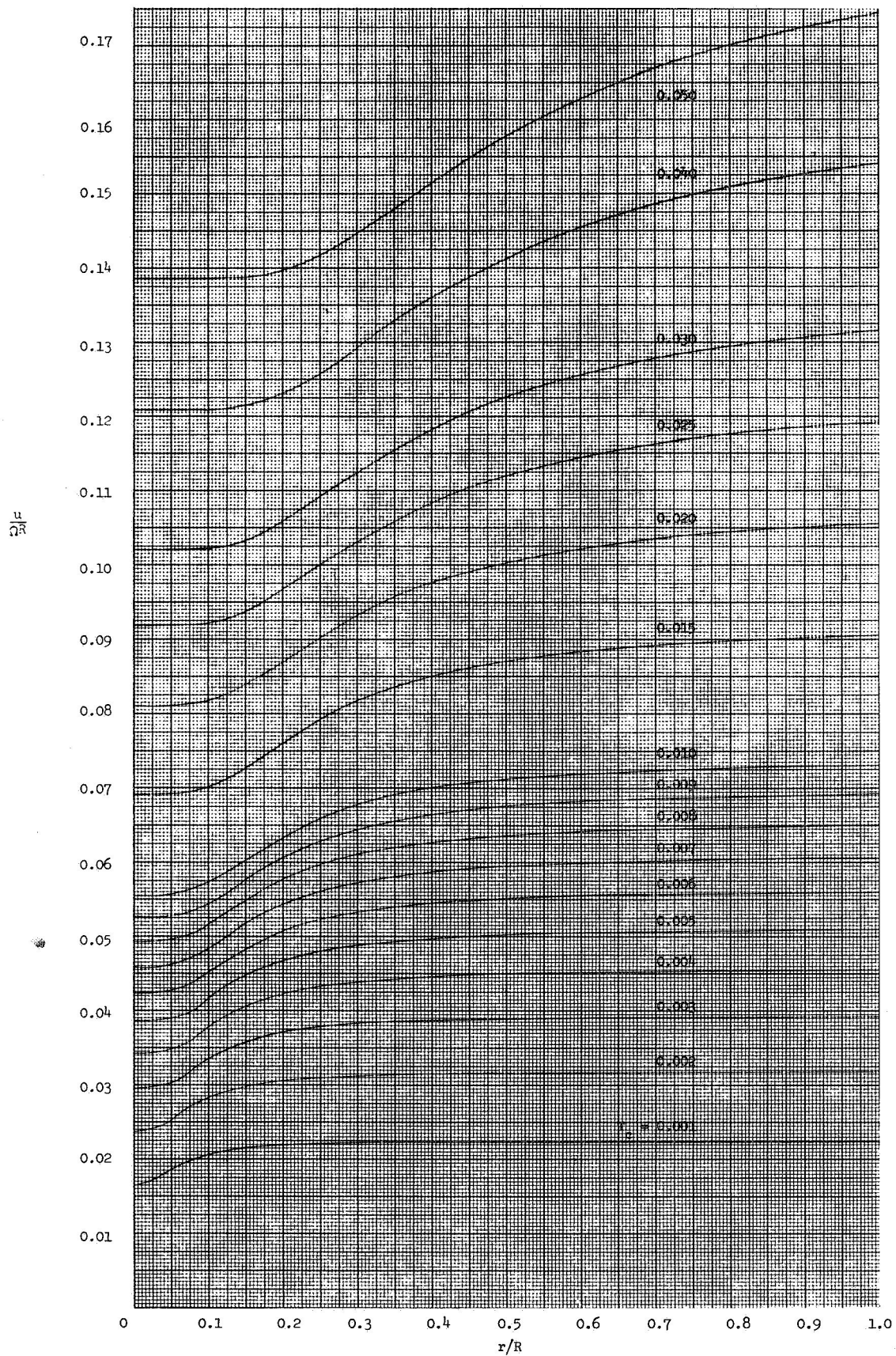


Figure 3.- Optimum radial distribution of axial inflow velocity for various values of thrust coefficient.

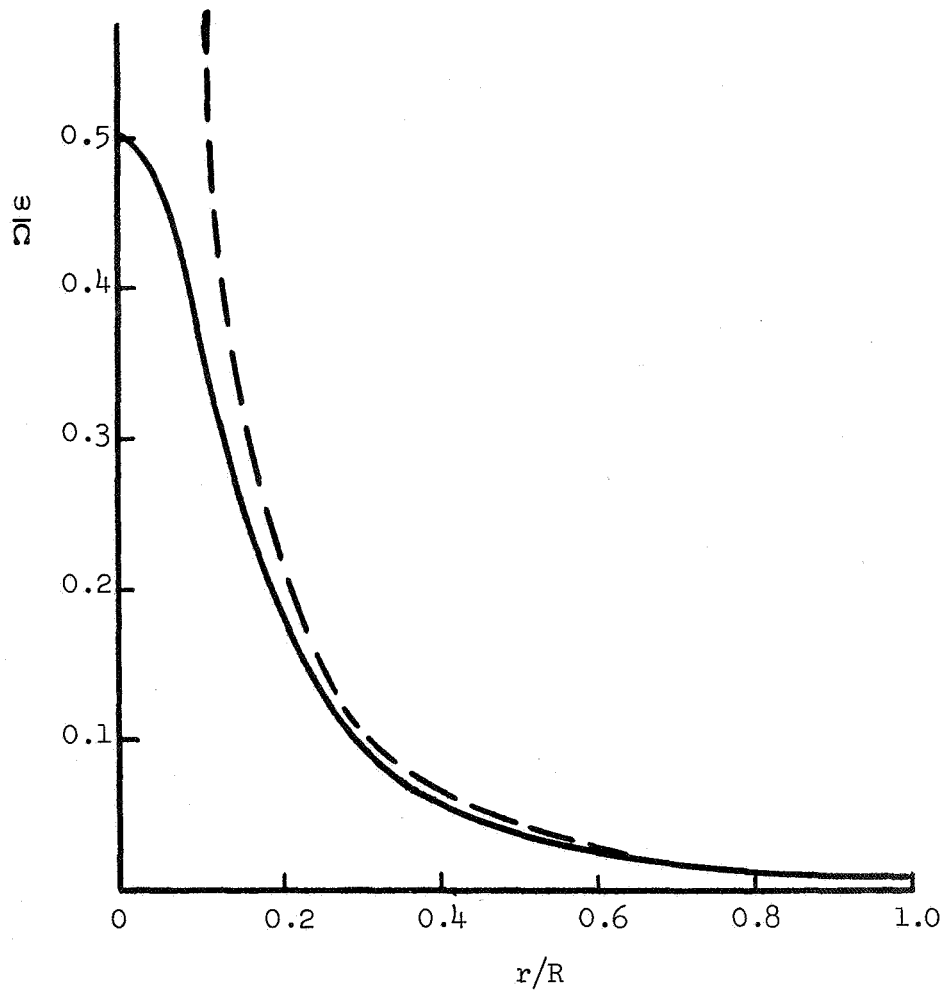


Figure 4.- Comparison of approximate solution with present result - optimum angular velocity distribution for  $T_c = 0.010$ .

— — — approximate solution (Ref. 5)  
 ————— present result



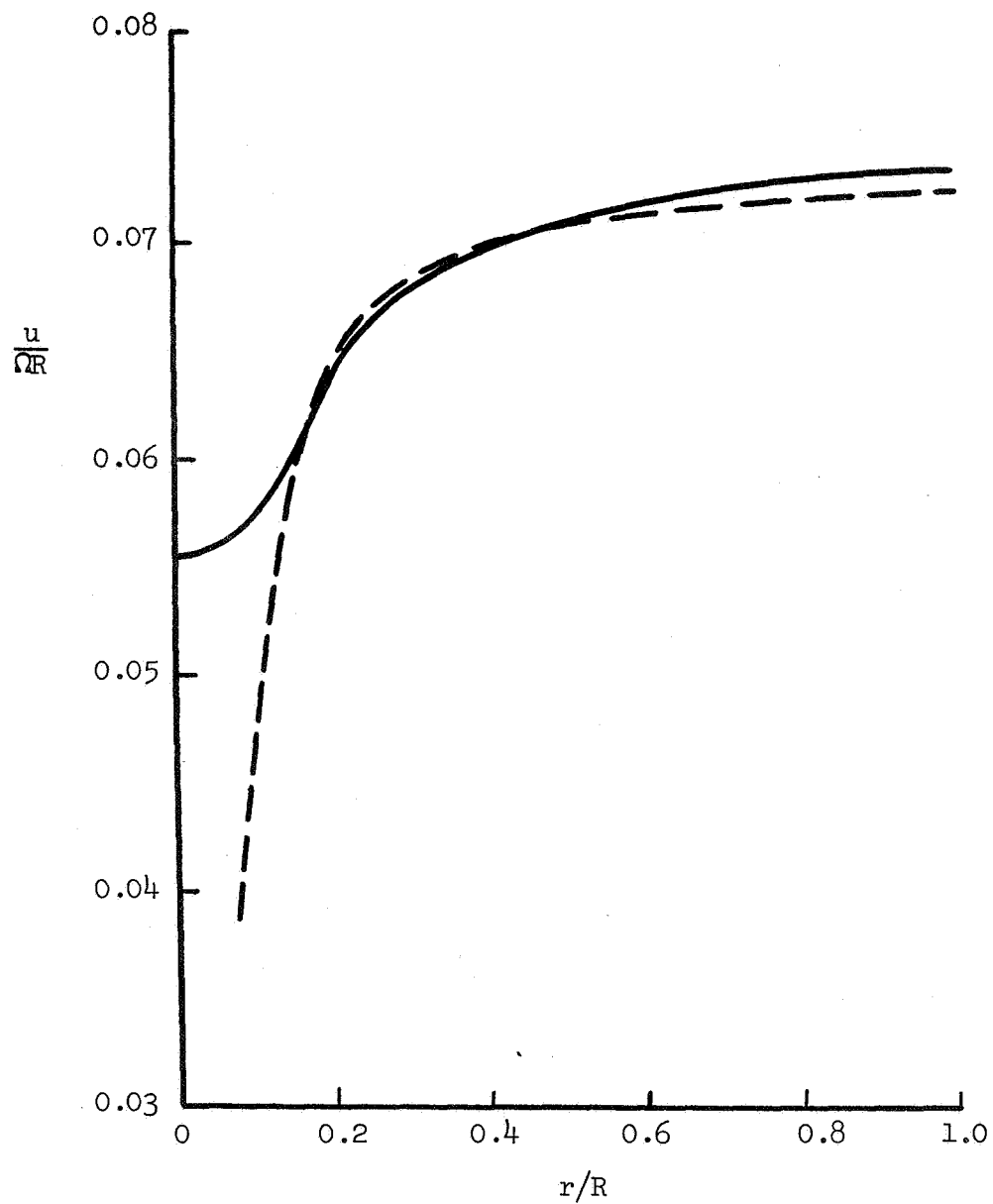


Figure 5.- Comparison of approximate solution with present result - optimum axial inflow velocity distribution for  $T_c = 0.010$ .

--- approximate solution (Ref. 5)  
 — present result

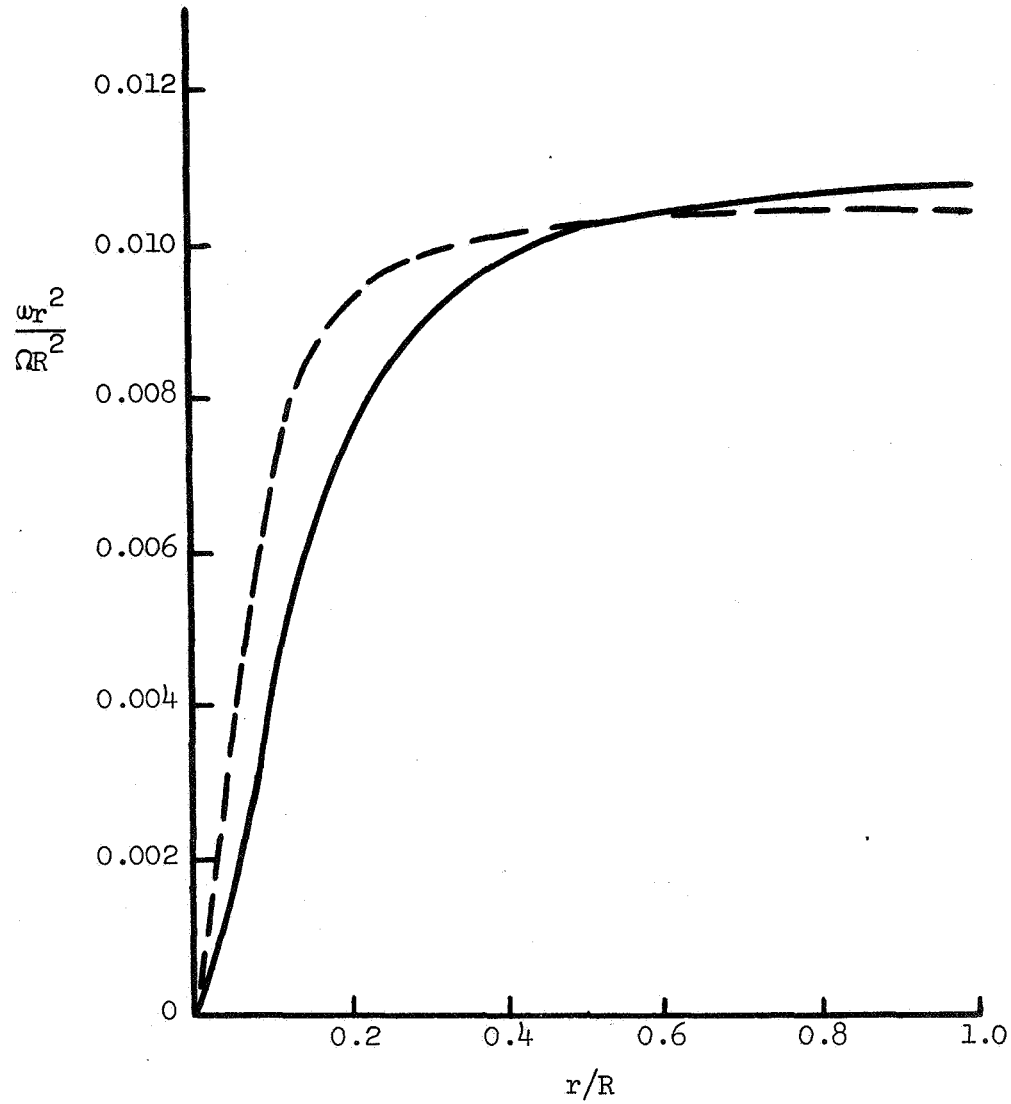


Figure 6.- Comparison of approximate solution with present result - optimum circulation distribution for  $T_c = 0.010$ .

- - - - - approximate solution (Ref. 5)  
 ————— present result

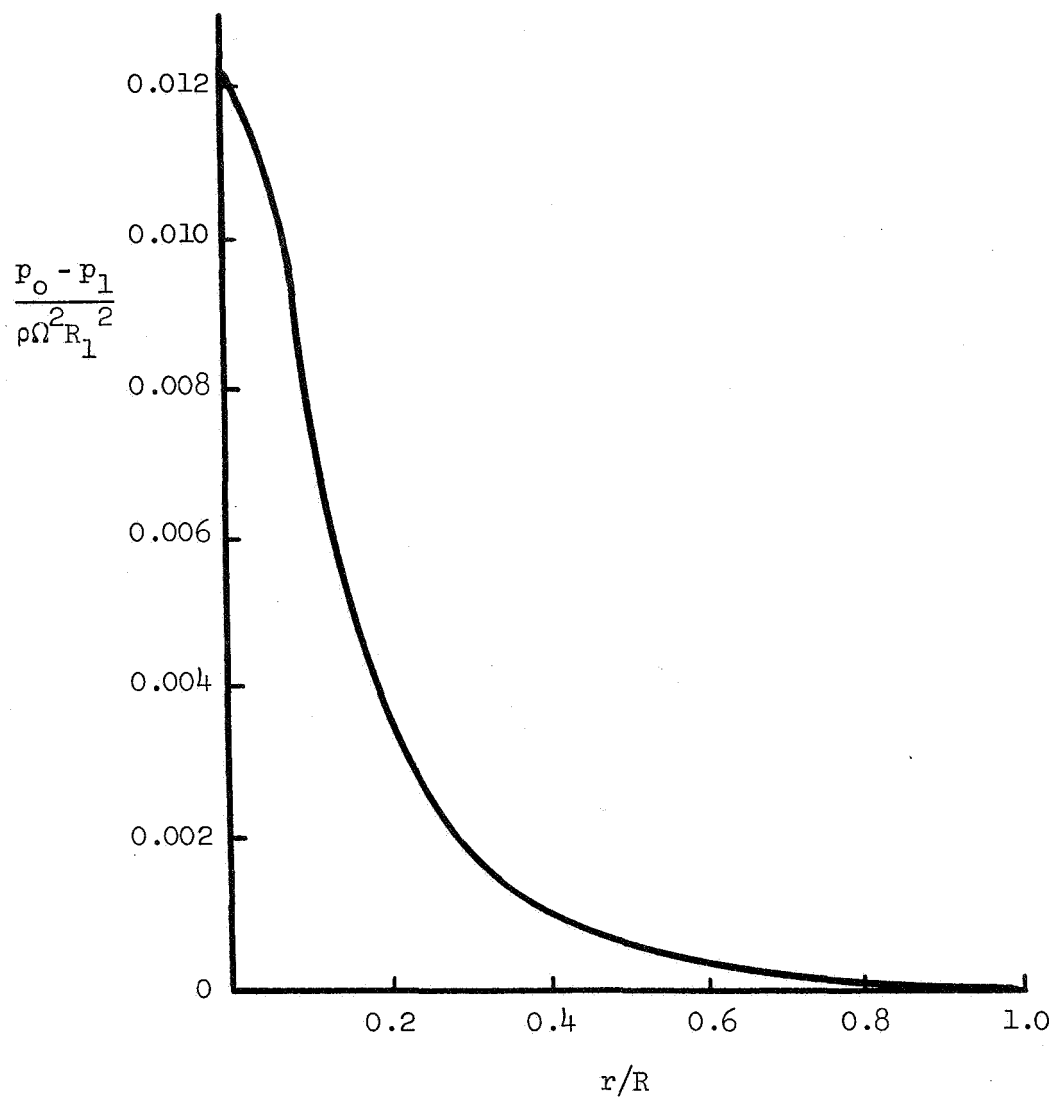


Figure 7.- Radial pressure distribution in the ultimate wake for  $T_c = 0.010$ .

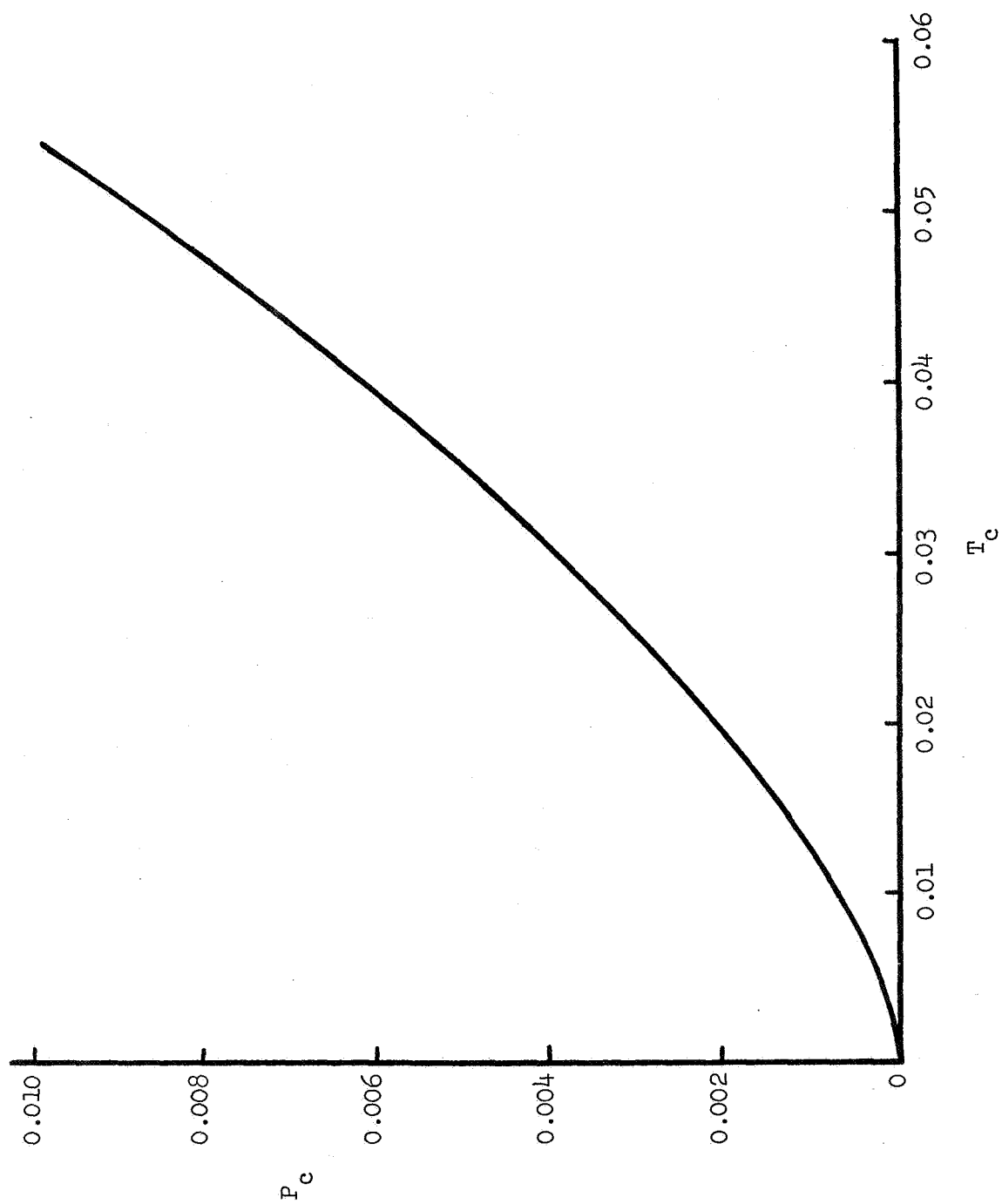


Figure 8.- Optimum power coefficient vs. thrust coefficient.

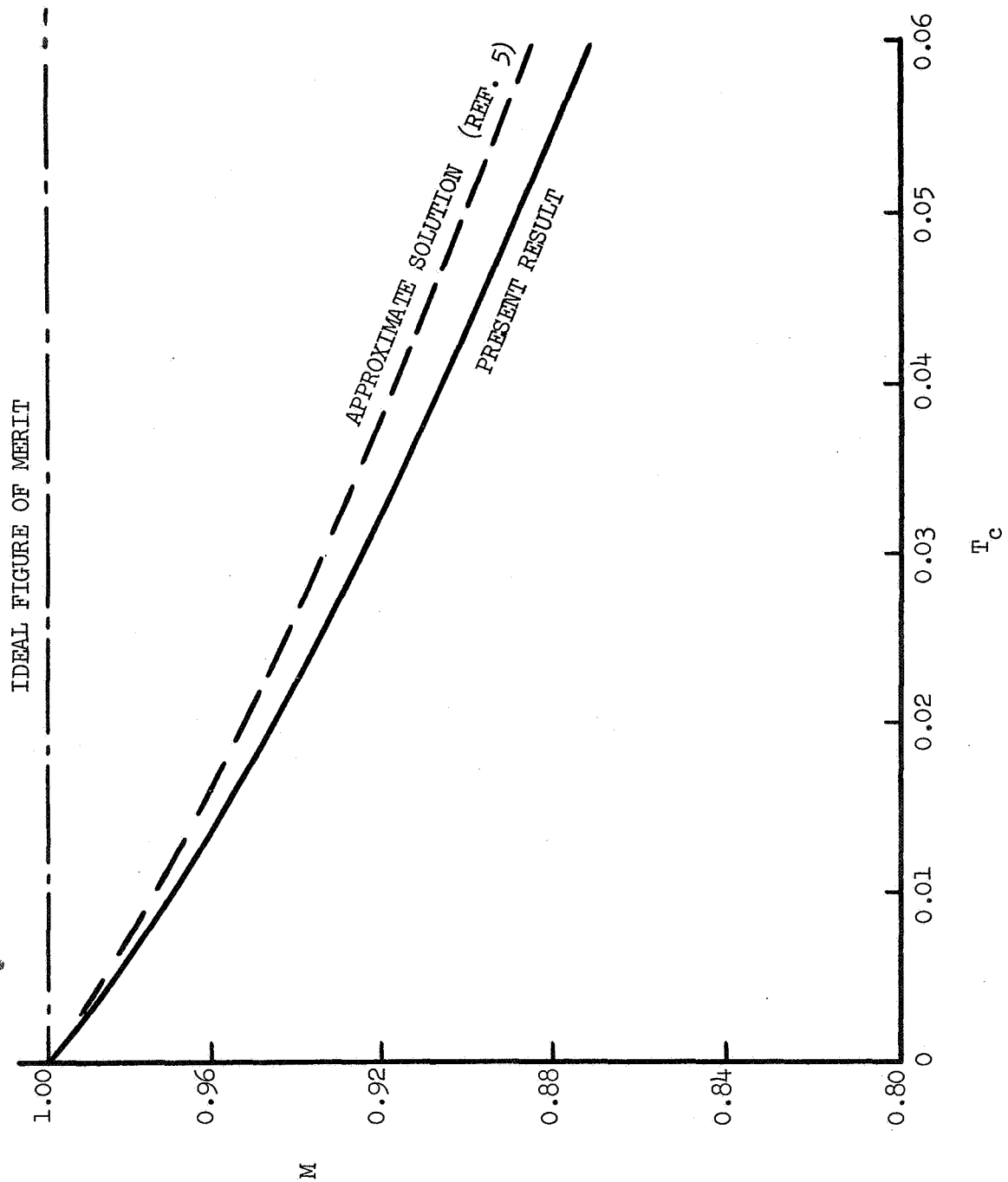


Figure 9.- Optimum figure of merit vs. thrust coefficient.

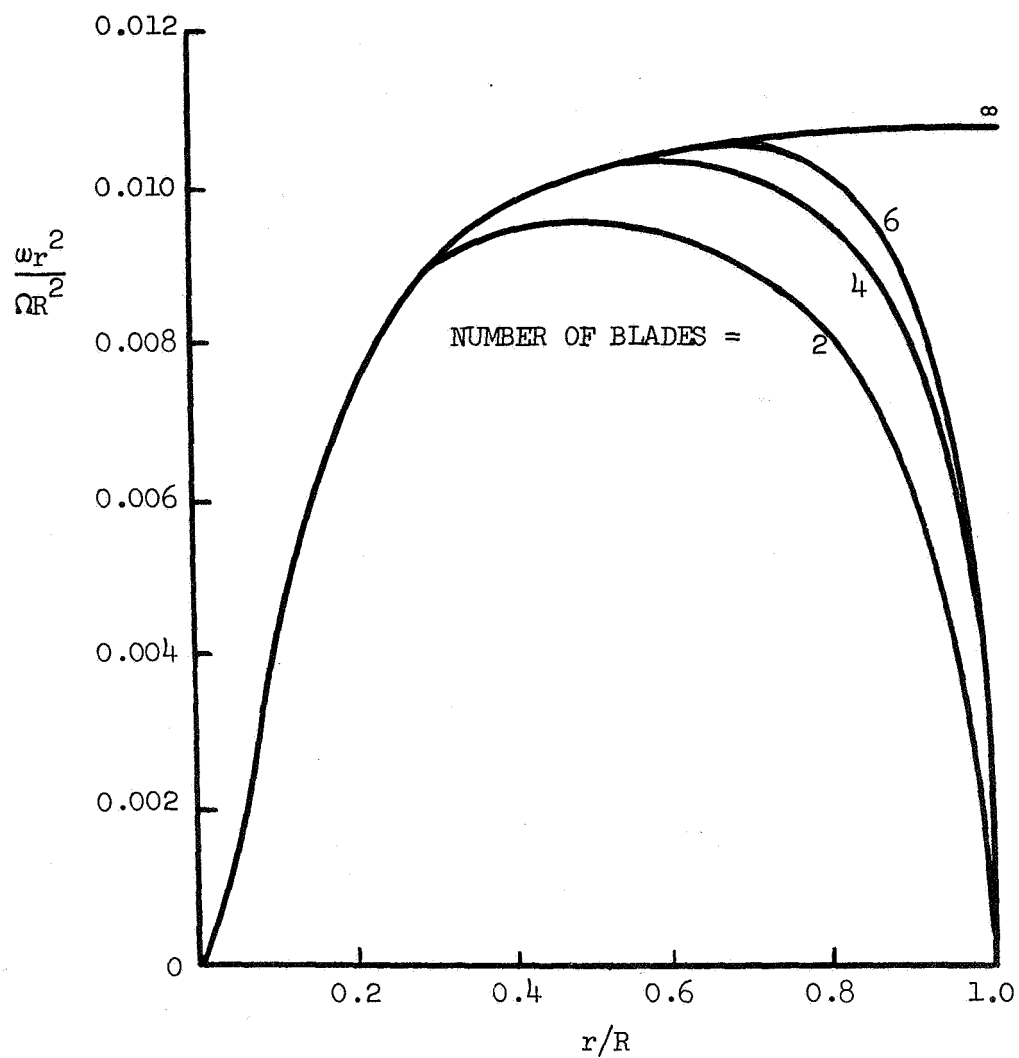


Figure 10.- Effect of number of blades on the distribution of circulation -  $T_c = 0.010$ .